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2000 J. Phys. A: Math. Gen. 33 8319

(http://iopscience.iop.org/0305-4470/33/46/312)

J. Phys. A: Math. Gen. **33** (2000) 8319–8330. Printed in the UK PII: S0305-4470(00)14544-5

On cyclic symmetries of n-dimensional nonlinear wave equations

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Received 8 June 2000, in final form 29 August 2000

Abstract. We consider the three n-dimensional nonlinear wave equations

$$u_{tt} = \sum_{k=1}^{n}[F_k(u)u_{x_k}]_{x_k} \qquad u_{tt} = \sum_{k=1}^{n} F_k(u_{x_k})u_{x_kx_k} \qquad \text{and} \qquad u_{tt} = \sum_{k=1}^{n} F_k(u_{x_kx_k}).$$

We also consider a special class of point transformations. Motivated by the results on the corresponding one-dimensional equations we present a class of discrete symmetries for these equations. In some cases these discrete symmetries form cyclic groups of finite order. Furthermore, point transformations exist that relate different equations but of the same class. The equivalence point transformations for each of the above general equations are presented.

1. Introduction

Transformation properties of one-dimensional nonlinear wave equations have been widely studied because of the many practical benefits that such knowledge gives and also because of the variety of physical applications for which these equations are model equations. Probably the most useful point transformations of partial differential equations (PDEs) are those which form a continuous (Lie) group of transformations, each member of which leaves an equation invariant. These may be employed to derive new solutions directly or via similarity reductions. Ibragimov [1] and Fushchich *et al* [2] provide two excellent sources of reference of such transformations of a large number of PDEs as well as for their many and varied physical applications. The Lie point transformations of the most common classes of one-dimensional nonlinear wave equations, $u_{tt} = [F(u)u_x]_x$, $u_{tt} = F(u_x)u_{xx}$ and $u_{tt} = F(u_{xx})$ have been investigated in [3], [4] and [5]† respectively.

In addition to possessing continuous groups of symmetries many PDEs also possess *discrete point symmetries*, which contribute to the full symmetry group. Such an example is given by Kingston and Sophocleous [6] who found that the reciprocal point transformation (double application gives the identity transformation) $x' = x/t$, $t' = 1/t$, $u' = -ut + x$ leaves the Burger-type equation $u_t + uu_x + (f(t) - f(1/t))u_{xx} = 0$ invariant, which is a symmetry additional to the Lie point symmetries obtained from the classical approach [7]. Furthermore PDEs admit *equivalence transformations*. These are transformations which have the property to transform any member of a class of PDEs to a PDE which is also a member of the class.

The discrete point symmetries for the classes of one-dimensional nonlinear wave equations $u_{tt} = [F(u)u_x]_x$, $u_{tt} = F(u_x)u_{xx}$ and $u_{tt} = F(u_{xx})$ have been classified in [8]. It turns out

† See [2, p 212].

that these discrete symmetries form cyclic groups of finite order. This paper is in the spirit of this later work. We generalize the results of [8] to the n-dimensional nonlinear wave equations

$$u_{tt} = \sum_{k=1}^{n} [F_k(u)u_{x_k}]_{x_k} \tag{1.1}$$

$$u_{tt} = \sum_{k=1}^{n} F_k(u_{x_k})u_{x_k x_k} \tag{1.2}$$

$$u_{tt} = \sum_{k=1}^{n} F_k(u_{x_k x_k}). \tag{1.3}$$

Lie symmetries of equation (1.1) when $n = 2$ and $n = 3$ have been studied [1].

In the following section we introduce the restricted classes of point transformations which are employed to construct the desired results. Furthermore, some preliminary results are derived for the transformed derivatives of u. Sections 3–5 present the point transformations obtained for the PDEs (1.1)–(1.3), respectively. In section 3 we give a detailed derivation of how these results are obtained, while in sections 4 and 5 we only state the results. Finally, in section 6 we give some final comments.

2. The class of point transformations: basic results

We consider the point transformations of the general class

$$x_i' = P_i(\boldsymbol{x}, t, u) \qquad t' = Q(\boldsymbol{x}, t, u) \qquad u' = R(\boldsymbol{x}, t, u) \qquad i = 1, 2, \ldots, n \tag{2.1}$$

where $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$, relating $x_1, \ldots, x_n, t, u(x_1, \ldots, x_n, t)$ and $x_1', \ldots, x_n', t'$, $u'(x_1', \ldots, x_n', t')$. We assume that these are non-degenerate in the sense that the Jacobian

$$J = \frac{\partial(P_1, P_2, \ldots, P_n, Q, R)}{\partial(x_1, x_2, \ldots, x_n, t, u)} \neq 0 \tag{2.2}$$

and also that

$$\delta = \frac{\partial(P_1, P_2, \ldots, P_n, Q)}{\partial(x_1, x_2, \ldots, x_n, t)} \neq 0. \tag{2.3}$$

In (2.3) P_i and Q are regarded as functions of $x_1, \ldots, x_n$ and t, using the fact that $u = u(x_1, \ldots, x_n, t)$, whereas in (2.2) P_i, Q and R are regarded as functions of the $n+2$ independent variables $x_1, \ldots, x_n, t$ and u.

In [9] a detailed derivation for the derivatives of u' in terms of the derivatives of u, in the case where u (and u') depends only on two independent variables, is presented. Here we generalize these results to the case where u (and u') depends on $n + 1$ independent variables by stating two formulas, for the first and second derivatives, respectively. The first derivatives of u', in terms of the derivatives of u, are given by the formula

$$\boldsymbol{U_1} = \boldsymbol{M}\boldsymbol{\Delta}^{-1} \tag{2.4}$$

where $\boldsymbol{U_1}$, $\boldsymbol{M}$ and $\boldsymbol{\Delta}$ are $1 \times (n+1)$, $1 \times (n+1)$ and $(n+1) \times (n+1)$ matrices, respectively, defined as follows:

$$\boldsymbol{U_1} = [u'_{x_1'}, u'_{x_2'}, \ldots, u'_{x_n'}, u'_{t'}] \tag{2.5}$$

$$\boldsymbol{M} = [D_{x_1}R, D_{x_2}R, \ldots, D_{x_n}R, D_t R] \tag{2.6}$$

$$\mathbf{\Delta} = \begin{bmatrix} D_{x_1}P_1 & D_{x_2}P_1 & \cdots & D_{x_n}P_1 & D_tP_1 \\ D_{x_1}P_2 & D_{x_2}P_2 & \cdots & D_{x_n}P_2 & D_tP_2 \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ D_{x_1}P_n & D_{x_2}P_n & \cdots & D_{x_n}P_n & D_tP_n \\ D_{x_1}Q & D_{x_2}Q & \cdots & D_{x_n}Q & D_tQ \end{bmatrix} \tag{2.7}$$

where D_k is the total derivative with respect to the indicated variable. We note that the inverse of the matrix $\mathbf{\Delta}$ exists, since its determinant is given by equation (2.3). The second derivatives can be found from the formula

$$\boldsymbol{U_2} = \boldsymbol{N}\mathbf{\Delta}^{-1} \tag{2.8}$$

where $\boldsymbol{U_2}$ and $\boldsymbol{N}$ are $(n+1) \times (n+1)$ matrices defined as follows:

$$\boldsymbol{U_2} = \begin{bmatrix} u'_{x'_1x'_1} & u'_{x'_1x'_2} & \cdots & u'_{x'_1x'_n} & u'_{x'_1t'} \\ u'_{x'_2x'_1} & u'_{x'_2x'_2} & \cdots & u'_{x'_2x'_n} & u'_{x'_2t'} \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ u'_{x'_nx'_1} & u'_{x'_nx'_2} & \cdots & u'_{x'_nx'_n} & u'_{x'_nt'} \\ u'_{t'x'_1} & u'_{t'x'_2} & \cdots & u'_{t'x'_n} & u'_{t't'} \end{bmatrix} \tag{2.9}$$

$$\boldsymbol{N} = \begin{bmatrix} D_{x_1}u'_{x'_1} & D_{x_2}u'_{x'_1} & \cdots & D_{x_n}u'_{x'_1} & D_tu'_{x'_1} \\ D_{x_1}u'_{x'_2} & D_{x_2}u'_{x'_2} & \cdots & D_{x_n}u'_{x'_2} & D_tu'_{x'_2} \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ D_{x_1}u'_{x'_n} & D_{x_2}u'_{x'_n} & \cdots & D_{x_n}u'_{x'_n} & D_tu'_{x'_n} \\ D_{x_1}u'_{t'} & D_{x_2}u'_{t'} & \cdots & D_{x_n}u'_{t'} & D_tu'_{t'} \end{bmatrix}. \tag{2.10}$$

In this paper we introduce two special classes of point transformations, namely

$$x'_i = P_i(x_i), \qquad t' = Q(t) \qquad u' = R(x_1, x_2, \ldots, x_n, t, u) \qquad i = 1, 2, \ldots, n \tag{2.11}$$

$$\begin{aligned} &x'_1 = P_1(t) \qquad x'_i = P_i(x_i) \qquad t' = Q(x_1) \\ &u' = R(x_1, x_2, \ldots, x_n, t, u) \qquad i = 2, 3, \ldots, n. \end{aligned} \tag{2.12}$$

We note that conditions (2.2) and (2.3) are satisfied if we assume that $P_1, P_2, \ldots, P_n, Q$ are not constant functions and also that $R_u \neq 0$. Employment of these point transformations lead to simple forms for the derivatives of u'. For the point transformations (2.11) the first derivatives of u', given by equation (2.4), simplify to

$$u'_{x'_i} = P^{-1}_{ix_i}(R_{x_i} + R_u u_{x_i}) \qquad i = 1, 2, \ldots, n \tag{2.13}$$

$$u'_{t'} = Q^{-1}_t(R_t + R_u u_t) \tag{2.14}$$

and the second-order pure derivatives, given by equation (2.8), simplify to

$$\begin{aligned} u'_{x'_ix'_i} = P^{-3}_{ix_i}[&P_{ix_i}R_u u_{x_ix_i} + P_{ix_i}R_{uu}u^2_{x_i} + (2P_{ix_i}R_{ux_i} - P_{ix_ix_i}R_u)u_{x_i} \\ &+ P_{ix_i}R_{x_ix_i} - P_{ix_ix_i}R_{x_i}] \qquad i = 1, 2, \ldots, n \end{aligned} \tag{2.15}$$

$$u'_{t't'} = Q^{-3}_t[Q_tR_uu_{tt} + Q_tR_{uu}u^2_t + (2Q_tR_{ut} - Q_{tt}R_u)u_t + Q_tR_{tt} - Q_{tt}R_t]. \tag{2.16}$$

Similar results may be obtained when we use point transformations (2.12). For example,

$$u'_{t'} = Q^{-1}_{x_1}(R_{x_1} + R_u u_{x_1}) \tag{2.17}$$

$$\begin{aligned} u'_{t't'} = Q^{-3}_{x_1}[&Q_{x_1}R_u u_{x_1x_1} + Q_{x_1}R_{uu}u^2_{x_1} + (2Q_tR_{ux_1} - Q_{x_1x_1}R_u)u_{x_1} \\ &+ Q_{x_1}R_{x_1x_1} - Q_{x_1x_1}R_{x_1}]. \end{aligned} \tag{2.18}$$

In the subsequent analysis we search for point transformations of the forms (2.11) and (2.12) for the three n-dimensional PDEs (1.1)–(1.3). For each of these PDEs we split the analysis into two cases. In the first case the special class of point transformations (2.11) is employed, while in the second we use (2.12).

3. Equation $u_{tt} = \sum_{k=1}^{n}[F_k(u)u_{x_k}]_{x_k}$

Case 1. We consider the point transformations (2.11) which relate the two PDEs

$$u'_{t't'} = \sum_{k=1}^{n}[G_k(u')u'_{x'_k}]_{x'_k} \tag{3.1}$$

$$u_{tt} = \sum_{k=1}^{n}[F_k(u)u_{x_k}]_{x_k} \tag{3.2}$$

where F_i are not all constant functions. In particular, these transformations will be symmetries of equation (3.2) if $G_i = F_i$, for all i.

Using equations (2.13)–(2.16), PDE (3.1) becomes an identity of the form

$$E_1(\boldsymbol{x}, t, u, u_{x_1}, \ldots, u_{x_n}, u_t, u_{x_1x_1}, \ldots, u_{x_nx_n}) = 0 \tag{3.3}$$

where u_{tt} has been eliminated from equation (3.2). The function E_1 is an explicit polynomial in $u_{x_1}, \ldots, u_{x_n}, u_t, u_{x_1x_1}, \ldots, u_{x_nx_n}$. We impose the condition that equation (3.3) is an identity in the variables that E_1 depends on which are regarded as independent. Thus, identity (3.3) produces a set of determining equations corresponding to the coefficients of different terms in $u_{x_1}, \ldots, u_{x_n}, u_t, u_{x_1x_1}, \ldots, u_{x_nx_n}$. These equations enable the desired point transformations to be derived and ultimately impose restrictions on the functional forms of $P_1,\ P_2, \ldots, P_n, Q$ and R. Also restrictions are made on the forms of the functions $F_i(u)$ and $G_i(u')$.

The coefficients of $u_{x_ix_i}$ give

$$G_i(u') = P_{ix_i}^2 Q_t^{-2} F_i(u) \qquad i = 1, 2, \ldots, n. \tag{3.4}$$

These latter relations also make the coefficients of $u_{x_i}^2$ vanish. Now coefficients of u_t^2 and u_t give, respectively, $Q_t^{-2}R_{uu} = 0$ and $Q_t^{-3}(2Q_tR_{ut} - Q_{tt}R_u) = 0$. Hence,

$$R = A(x_1, x_2, \ldots, x_n)Q_t^{1/2}u + B(x_1, x_2, \ldots, x_n, t). \tag{3.5}$$

Firstly, we assume that none of the functions $G_i(u')$ is constant. Differentiation of equation (3.4) with $i = 1$ w.r.t. x_j, $j = 2, 3, \ldots, n$, and with $i = 2$ w.r.t. x_1 give

$$R_{x_1} = R_{x_2} = \cdots = R_{x_n} = 0.$$

Therefore the form of R, given by equation (3.5), reduces to

$$R = cQ_t^{1/2}u + B(t) \tag{3.6}$$

where c is a constant.

Coefficients of u_{x_i} in equation (3.3) imply that the functions $P_i(x_i)$ are linear. That is,

$$P_i(x_i) = \alpha_{i1}x_i + \alpha_{i2} \qquad i = 1, 2, \ldots, n. \tag{3.7}$$

The relations (3.4), (3.6) and (3.7) simplify equation (3.3) to

$$\tfrac{1}{4}Q_t^{-7/2}(2Q_tQ_{ttt} - 3Q_{tt}^2)cu + Q_t^{-3}(Q_tB_{tt} - Q_{tt}B_t) = 0. \tag{3.8}$$

The coefficient of u in (3.8) shows that (i) $Q(t) = \beta_1t + \beta_2$ or (ii) $Q(t) = \frac{1}{\beta_1t+\beta_2} + \beta_3$.

For case (i), using the fact that Q is linear, then differentiation of equation (3.4), with $i = 1$, with respect to t implies that B must be constant, say γ_2, since G_1 is not constant. Hence, equation (3.6) gives $R = \gamma_1 u + \gamma_2$, where $\gamma_1 = c\sqrt{\beta_1}$. Therefore, we have the following point transformation:

$$x_i' = \alpha_{i1} x_i + \alpha_{i2} \qquad t' = \beta_1 t + \beta_2 \qquad u' = \gamma_1 u + \gamma_2 \qquad i = 1, 2, \ldots, n \tag{3.9}$$

where

$$F_i(u) = \beta_1^2 \alpha_{i1}^{-2} G_i(\gamma_1 u + \gamma_2) \tag{3.10}$$

which transforms equation (3.1) to (3.2). We note that transformation (3.9) is valid for arbitrary functions $F_i(u)$. Such transformations are known as *equivalence transformations*. These transformations may be filtered out of equations arising in the subsequent analysis, without loss of generality, with the understanding that all transformations obtained for equation (3.2) may be augmented by (3.9) and (3.10).

In case (ii), $Q(t) = \frac{\epsilon}{t}$, where $\epsilon = \pm 1$, modulo (3.9). From equation (3.8), $B = c_1 Q + c_2$. Hence, $B(t) = \frac{\epsilon c_1}{t} + c_2$. Equivalence transformation (3.9) implies that $c_1 = c_2 = 0, c\sqrt{-\epsilon} = 1$ (equation (3.6)) and $\alpha_{i1} = 1, \alpha_{i2} = 0$ (equation (3.7)), without loss of generality. That is, we generate the point transformation

$$x_i' = x_i \qquad t' = \frac{\epsilon}{t} \qquad u' = \frac{u}{t} \qquad i = 1, 2, \ldots, n. \tag{3.11}$$

Transformation (3.11) forms a cyclic group of order two if $\epsilon = 1$ and of order four if $\epsilon = -1$. Now equations (3.4) provide us with the forms of $F_i(u)$ and $G_i(u')$. We differentiate equations (3.4) with respect to u and t to give, respectively, $G_{iu'} t^{-1} = t^4 F_{iu}$ and $-G_{iu'} u t^{-2} = 4t^3 F_i$. Eliminating $G_{iu'}$ from these equations we obtain

$$u^{-1} t^5 \left(u \frac{\mathrm{d}F_i}{\mathrm{d}u} + 4F_i \right) = 0.$$

Hence, $F_i(u) = u^{-4}$, where the constant of integration may be taken equal to 1 using (3.9). Equations (3.4) give $G_i(u') = \frac{t^4}{u^4} = u'^{-4}$. Therefore equation (3.11) is a discrete symmetry of the nonlinear wave equation

$$u_{tt} = \sum_{k=1}^{n} [u^{-4} u_{x_k}]_{x_k}. \tag{3.12}$$

Now we assume that $G_1(u')$ is a non-constant function and $G_i(u') = \mu_i, i = 2, 3, \ldots, n$, where μ_i are constants. Since the functions P_i and Q do not depend on u, from equations (3.4) we deduce that the functions $F_i(u), i = 2, 3, \ldots, n$, must also be constants. Also, from the same equations we conclude that $P_i(x_i), i = 2, 3, \ldots, n$ and $Q(t)$ are linear functions. Therefore, $P_i = x_i, i = 2, 3, \ldots, n, Q = t$, modulo (3.9). Equations (3.4) give $F_i(u) = \mu_i, i = 2, 3, \ldots, n$. Furthermore, differentiation of the first equation of equations (3.4) with respect to $x_2, x_3, \ldots, x_n, t$ give, respectively, $R_{x_2} = R_{x_3} = \cdots = R_{x_n} = R_t = 0$. Hence, the form of R, given by equation (3.5), simplifies to

$$R = A(x_1) u + B(x_1). \tag{3.13}$$

Using the forms of P_i, Q and R, the coefficient of u_{x_1} in equation (3.3) give

$$P_{1x_1}^{-1} \left(-2A_{x_1} P_{1x_1} u \frac{\mathrm{d}F_1}{\mathrm{d}u} - 2B_{x_1} P_{1x_1} \frac{\mathrm{d}F_1}{\mathrm{d}u} + (A P_{1x_1x_1} - 2A_{x_1} P_{1x_1}) F_1 \right) = 0. \tag{3.14}$$

From identity (3.14) we deduce that, either F_1, $\frac{\mathrm{d}F_1}{\mathrm{d}u}$ and $u\frac{\mathrm{d}F_1}{\mathrm{d}u}$ are linearly related with constant coefficients or all three constant coefficients are zero. Thus, we have (i) $P_1(x_1) = x_1$, $A(x_1)$ and $B(x_1)$ being constants and $F_1(u)$ an arbitrary function, (ii) $F_1(u) = \mathrm{e}^u$, or (iii) $F_1(u) = u^m$, where m is a constant. In all three subcases the linear transformations (3.9) have been used to filter out arbitrary constants.

Clearly, for subcase (i) we arrive at the equivalence transformations (3.9). Similarly, without giving any derivations, we state that subcase (ii) yields no point transformations other than those of equation (3.9).

In subcase (iii), where $F_1 = u^m$, equation (3.14) gives $B = \text{constant}$, which may be taken equal to zero because of the transformations (3.9), and $A(x_1) = c_1 P_{x_1}^{1/(2(m+1))}$, where c_1 is a constant. Now, equation (3.3) implies that $2P_{1x_1}P_{1x_1x_1x_1} - 3P_{1x_1x_1}^2 = 0$. Since $P_1(x_1)$ is not linear, otherwise we are led to transformations (3.9), we have $P_1(x_1) = \frac{\epsilon}{x_1}$, where $\epsilon = \pm 1$. The transformations (3.9) have also been used here to eliminate constants of integration in the form of $P_1(x_1)$. The first equation in equations (3.4) takes the form $G_1(u') = P_{1x_1}^2 u^m$. We differentiate this equation with respect to x_1 and u, respectively, and then we eliminate $G_{1u'}$ from the two resulting expressions to give

$$c_1^{-1} P_{1x_1}^{(4m+3)/(2m+2)} u^{m-1}(-3m-4) = 0.$$

Hence, $m = -\frac{4}{3}$. Therefore we have $F_1(u) = u^{-4/3}$, $R = A(x_1)u = c_1 P_{x_1}^{-3/2} u = x_1^3 u$, where the constant has been filtered out, and $G_1(u') = x_1^{-4}u^{-4/3} = u'^{-4/3}$. Thus, we have produced the discrete cyclic symmetry

$$x_1' = \frac{\epsilon}{x_1} \qquad x_i' = x_i \qquad t' = t \qquad u' = x_1^3 u \qquad i = 2, 3, \ldots, n \quad (3.15)$$

which leaves invariant the nonlinear PDE

$$u_{tt} = [u^{-4/3}u_{x_1}]_{x_1} + \sum_{k=2}^{n} \mu_k u_{x_k x_k}. \quad (3.16)$$

We state, without giving any detailed derivations, that if more than one of the functions $G_i(u')$ are non-constants and the remaining are constants, then we are led to contradiction or to the equivalence transformation (3.9).

Case 2. Here we consider point transformations of the form (2.12) which transform equation (3.1) to (3.2). As in the previous case we substitute the forms of the derivatives of the transformed variables into equation (3.1) to obtain an identity similar to (3.3):

$$E_2(\boldsymbol{x}, t, u, u_{x_1}, \ldots, u_{x_n}, u_t, u_{x_1x_1}, \ldots, u_{x_nx_n}) = 0. \quad (3.17)$$

Coefficients of $u_{x_1x_1}$ and $u_{x_ix_i}$, $(i = 2, 3, \ldots, n)$ in (3.17) give, respectively,

$$G_1(u') = \frac{P_{1t}^2 Q_{x_1}^{-2}}{F_1(u)} \quad (3.18)$$

$$G_i(u') = -\frac{P_{ix_i}^2 Q_{x_1}^{-2} F_i(u)}{F_1(u)} \qquad i = 2, 3, \ldots, n. \quad (3.19)$$

Firstly, we assume that $F_1(u)$ is not a constant, which also implies that $G_1(u')$ is not a constant. Differentiation of (3.18) with respect to x_i, $(i = 2, 3, \ldots, n)$, respectively and differentiation of any of the relations (3.19) with respect to t, leads to $R = R(x_1, u)$. Furthermore, differentiation of (3.18) w.r.t. t and of (3.19) w.r.t. x_i $(i = 2, 3, \ldots n)$ give that $P_1(t)$, $P_2(x_2)$, $P_3(x_3), \ldots, P_n(x_n)$ are linear functions. That is, without loss of generality, $P_1 = t$ and $P_i = x_i$, $(i = 2, 3, \ldots, n)$.

Now coefficients of u_{x_1} and $u_{x_1}^2$ in (3.17) give, respectively,

$$R(x_1, u) = Q_{x_1}^{1/2}\phi(u) \quad (3.20)$$

$$F_1(u) = \frac{d\phi}{du}. \quad (3.21)$$

These results simplify (3.17) to

$$\tfrac{1}{4}Q_{x_1}^{-7/2}\phi(2Q_{x_1}Q_{x_1x_1x_1} - 3Q_{x_1x_1}^2) = 0.$$

Therefore we have either (i) $Q(x_1) = \beta_1 x_1 + \beta_2$ or (ii) $Q(x_1) = \frac{1}{\beta_1 x_1 + \beta_2} + \beta_3$.

For case (i) we take $Q = x_1$. Using the forms of $P_i(x_i)$ which have been obtained earlier and the relations (3.18) and (3.19) we deduce that the transformation

$$x_1' = t \qquad x_i' = x_i \qquad t' = x_1 \qquad u' = \phi(u) \qquad i = 2, 3, \ldots, n \tag{3.22}$$

transforms

$$u'_{t't'} = [\phi'(\phi^{-1}(u'))^{-1}u'_{x_1'}]_{x_1'} - \sum_{k=2}^{n}[\phi'(\phi^{-1}(u'))^{-1}F_k(\phi^{-1}(u'))u'_{x_k'}]_{x_k'}$$

to

$$u_{tt} = [\phi'(u)u_{x_1}]_{x_1} + \sum_{k=2}^{n}[F_k(u)u_{x_k}]_{x_k}$$

where ϕ and F_i are arbitrary functions. We note that (3.22) forms a cyclic group of order 2 if the function $\phi(u)$ is equal to its inverse. We present special cases of the transformation (3.22) by choosing specific forms of $\phi(u)$.

Example 1. The transformation

$$x_1' = t \qquad x_i' = x_i \qquad t' = x_1 \qquad u' = u^m \qquad i = 2, 3, \ldots, n$$

transforms the PDE

$$u'_{t't'} = \frac{1}{m}[u'^{(1-m)/m}u'_{x_1'}]_{x_1'} - \frac{1}{m}\sum_{k=2}^{n}[u'^{(1-m)/m}F_k(u'^{1/m})u'_{x_k'}]_{x_k'}$$

to the PDE

$$u_{tt} = m[u^{m-1}u_{x_1}]_{x_1} + \sum_{k=2}^{n}[F_k(u)u_{x_k}]_{x_k}.$$

Symmetry within this example occurs when $m = -1$ and $F_k(u) = u^{-1}$. That is, the point transformation

$$x_1' = t \qquad x_i' = x_i \qquad t' = x_1 \qquad u' = u^{-1} \qquad i = 2, 3, \ldots, n$$

is a cyclic symmetry of order 2 for the PDE

$$u_{tt} = -[u^{-2}u_{x_1}]_{x_1} + \sum_{k=2}^{n}[u^{-1}u_{x_k}]_{x_k}.$$

Also in the following example the point transformation that occurs is a symmetry under a certain condition.

Example 2. The transformation

$$x_1' = t \qquad x_i' = x_i \qquad t' = x_1 \qquad u' = -u + L \qquad i = 2, 3, \ldots, n$$

is a symmetry of the PDE

$$u_{tt} = -u_{x_1x_1} + \sum_{k=2}^{n}[F_k(u)u_{x_k}]_{x_k}$$

provided that the functions $F_k(u)$ are odd and periodic with period L.

Example 3. The transformation

$$x_1' = t \qquad x_i' = x_i \qquad t' = x_1 \qquad u' = \ln u \qquad i = 2, 3, \dots, n$$

transforms the PDE

$$u'_{t't'} = [\mathrm{e}^{u'} u'_{x_1'}]_{x_1'} - \sum_{k=2}^{n} [\mathrm{e}^{u'} F_k(\mathrm{e}^{u'}) u'_{x_k'}]_{x_k'}$$

to the PDE

$$u_{tt} = [u^{-1} u_{x_1}]_{x_1} + \sum_{k=2}^{n} [F_k(u) u_{x_k}]_{x_k}.$$

One of the main applications of the point transformations is to map a solution of a PDE to another solution of the same or different PDE. For example, we consider the solution $u(x_1, x_2, t) = t^2(\sin x_2 + c\mathrm{e}^{x_1})^{-2}$ of the PDE $u_{tt} = (u^{-1}u_{x_1})_{x_1} + (u^{-1}u_{x_2})_{x_2}$ [1, p 234]. Then employing the point transformation given in example 3, with $n = 2$ and $F_2(u) = u^{-1}$, we obtain the solution $u'(x_1', x_2', t') = \ln[x'^2_1(\sin x_2' + c\mathrm{e}^{t'})^2]$ of the PDE $u'_{t't'} = [\mathrm{e}^{u'} u'_{x_1'}]_{x_1'} - u'_{x_2' x_2'}$.

For case (ii) we take $Q = \frac{\epsilon}{x_1}$. Now we multiply the relation (3.18) w.r.t. u and x_1 to obtain, respectively, $G_{1u'} Q_{x_1}^{1/2} \phi_u = -Q_{x_1}^{-2} \phi_{uu} \phi_u^{-2}$ and $\frac{1}{2} G_{1u'} Q_{x_1}^{-1/2} Q_{x_1 x_1} \phi = -2Q_{x_1}^{-3} Q_{x_1 x_1} \phi_u^{-1}$. Elimination of $G_{1u'}$ from these two equations give

$$\phi\phi_{uu} - 4\phi_u^2 = 0. \tag{3.23}$$

Hence, $\phi = (\lambda_1 u + \lambda_2)^{-1/3}$. We take $\phi(u) = -3u^{-1/3}$ and from (3.20) we get $R = -3x_1^{-1}u^{-1/3}$. Also equation (3.18) gives $G_1(u') = x_1^4 u^{4/3} = u'^{-4}$. We differentiate equations (3.19) w.r.t. u and x_i $(i = 2, 3, \dots, n)$, respectively and then we eliminate $G_{iu'}$ from each of the corresponding pair of the resulting equations, to get

$$Q_{x_1}^{-5/2} \phi^{-1} \phi_u^{-3} [-\phi\phi_u F_{iu} + (\phi\phi_{uu} - 4\phi_u^2) F_i] = 0.$$

Clearly, equation (3.23) implies $F_{iu} = 0$. Hence, $F_i(u) = \mu_i, (i = 2, 3, \dots, n)$ and from (3.19) we have $G_i(u') = \mu_i x_1^4 u^{4/3} = \mu_i u'^{-4}$. Without loss of generality, $\mu_i = 1$.

Therefore, collecting all these results, we deduce that the transformation

$$x_1' = t \qquad x_i' = x_i \qquad t' = \epsilon x_1^{-1} \qquad u' = -3x_1^{-1}u^{-1/3}$$

transforms

$$u'_{t't'} = \sum_{k=1}^{n} [u'^{-4} u'_{x_k'}]_{x_k'}$$

to

$$u_{tt} = [u^{-4/3} u_{x_1}]_{x_1} + \sum_{k=2}^{n} u_{x_k x_k}.$$

Finally, we turn to the case where $F_1(u) = \mu_1$, where μ_1 is a constant. From (3.18) we have $P_1 = t, Q = x_1$ and $G_1(u') = \frac{1}{\mu_1}$. Furthermore, equations (3.19) produce the results $P_i = x_i, F_i(u) = \mu_i$ and $G_i(u') = -\frac{\mu_i}{\mu_1}$, where $i = 2, 3, \dots, n-1$. Coefficients of $u_{x_n}^2$ in (3.17) and equation (3.19) with $i = n$ give $R = P_{nx_n}^{2/m} u$ and $F_n(u) = u^m$. These results simplify the identity (3.17) to (ignoring nonzero factors)

$$(2mP_{nx_n} P_{nx_n x_n x_n} + 4P_{nx_n x_n}^2)u + mP_{nx_n} P_{nx_n x_n}(3m+4)u_{x_n} = 0.$$

The function $P_n(x_n)$ must not be linear, otherwise we obtain a special case of the transformation (3.22). In this latter equation, the coefficients of u and u_{x_n} must vanish. Hence,

$m = -\frac{4}{3}$ and, without loss of generality, $P_n = \frac{\epsilon}{x_n}$. Therefore $F_n(u) = u^{-4/3}$, $R(x_n, u) = x_n^3 u$ and from (3.19), with $i = n$, we get $G_n(u') = -\frac{1}{\mu_1} u'^{-4/3}$.

We collect these results and we choose $\mu_1 = -1$ so that the transformation

$$x_1' = t \qquad x_i' = x_i \qquad x_n' = \frac{\epsilon}{x_n} \qquad t' = x_1 \qquad u' = x_n^3 u \qquad i = 2, 3, \ldots, n-1$$

is a cyclic symmetry, of order 2 if $\epsilon = 1$ and of order 4 if $\epsilon = -1$, of the wave equation

$$u_{tt} = -u_{x_1 x_1} + \sum_{k=2}^{n-1} \mu_k u_{x_k x_k} + [u^{-4/3} u_{x_n}]_{x_n}.$$

4. Equation $u_{tt} = \sum_{k=1}^{n} F_k(u_{x_k}) u_{x_k x_k}$

We consider point transformations of the form (2.11) and (2.12) which relate the n-dimensional PDEs

$$u'_{t't'} = \sum_{k=1}^{n} G_k(u'_{x_k'}) u'_{x_k' x_k'} \tag{4.1}$$

$$u_{tt} = \sum_{k=1}^{n} F_k(u_{x_k}) u_{x_k x_k}. \tag{4.2}$$

Unlike the previous section, here we present the results without giving the detailed derivations. It turns out that no point transformations of the form (2.12) exist that transform (4.1) to (4.2).

The equivalence transformation that connects (4.1) and (4.2) is

$$\begin{aligned} &x_i' = \alpha_{i1} x_i + \alpha_{i2} \qquad t' = \beta_1 t + \beta_2 \\ &u' = \gamma u + \sum_{k=1}^{n} \gamma_k x_k + \gamma_{n+1} t + \delta \qquad i = 1, 2, \ldots, n \end{aligned} \tag{4.3}$$

where the relation between the functions $F_i(u_{x_i})$ and $G_i(u'_{x_i'})$ is given by

$$F_i(u_x) = \alpha_{i1}^{-2} \beta_1^2 G_i(\alpha_{i1}^{-1}(\gamma u_x + \gamma_i)). \tag{4.4}$$

Two examples of discrete symmetries are reported for equation (4.2). The transformation

$$x_i' = x_i \qquad t' = \frac{\epsilon}{t} \qquad u' = \frac{u}{t} \qquad i = 1, 2, \ldots, n \tag{4.5}$$

is a cyclic symmetry (of order 2 if $\epsilon = 1$ and of order 4 if $\epsilon = -1$) of the nonlinear equation

$$u_{tt} = \sum_{k=1}^{n} u_{x_k}^{-4} u_{x_k x_k}.$$

The second example is the transformation

$$\begin{aligned} &x_i' = x_i \qquad t' = t \qquad u' = u + \phi(x_m, x_{m+1}, \ldots, x_n) \\ &2 \leqslant m \leqslant n \qquad i = 1, 2, \ldots, n \end{aligned}$$

which is a symmetry of the PDE

$$u_{tt} = \sum_{k=1}^{m-1} F_k(u_{x_k}) u_{x_k x_k} + \sum_{k=m}^{n} \mu_k u_{x_k x_k}$$

where $F_k(u_{x_k})$ are arbitrary functions and $\phi(x_m, x_{m+1}, \ldots, x_n)$ satisfy the linear PDE

$$\phi_{tt} = \sum_{k=m}^{n} \mu_k \phi_{x_k x_k}.$$

5. Equation $u_{tt} = \sum_{k=1}^{n} F_k(u_{x_k x_k})$

Similarly, as in the previous section, without presenting any derivations we present the point transformations of the form (2.11) and (2.12) that map the equation

$$u'_{t't'} = \sum_{k=1}^{n} G_k(u'_{x'_k x'_k}) \tag{5.1}$$

into the equation

$$u_{tt} = \sum_{k=1}^{n} F_k(u_{x_k x_k}). \tag{5.2}$$

The equivalence transformation is

$$\begin{aligned} &x'_i = \alpha_{i1} x_i + \alpha_{i2} \qquad t' = \beta_1 t + \beta_2 \qquad i = 1, 2, \ldots, n \\ &u' = \gamma u + \sum_{i=1}^{n}\sum_{j=i}^{n} \gamma_{ij} x_i x_j + \sum_{i=1}^{n} \gamma_{in+1} x_i t + \gamma_{n+1n+1} t^2 + \sum_{i=1}^{n} \gamma_i x_i + \gamma_{n+1} t + \delta \end{aligned} \tag{5.3}$$

where the relation between the functions $F_k(u_{x_k x_k})$ and $G_k(u'_{x'_k x'_k})$ is given by

$$\begin{aligned} &F_1(u_{x_1 x_1}) = \gamma^{-1} \beta_1^2 G_1(\alpha_{11}^{-2}(\gamma u_{x_1 x_1} + 2\gamma_{11})) - 2\gamma_{n+1n+1}\gamma^{-1} \\ &F_i(u_{x_i x_i}) = \gamma^{-1} \beta_1^2 G_i(\alpha_{i1}^{-2}(\gamma u_{x_i x_i} + 2\gamma_{ii})) \qquad i = 2, 3, \ldots, n. \end{aligned} \tag{5.4}$$

Two additional transformations of the form (2.11) exist. The transformation

$$x'_i = x_i \qquad t' = \frac{\epsilon}{t} \qquad u' = \frac{u}{t} \qquad i = 1, 2, \ldots, n \tag{5.5}$$

is a symmetry of the PDE

$$u_{tt} = \sum_{k=1}^{n} (u_{x_k x_k})^{-3}$$

and the transformation

$$x'_1 = \frac{\epsilon}{x_1} \qquad x'_i = x_i \qquad t' = t \qquad u' = \frac{u}{x_1} \qquad i = 2, 3, \ldots, n \tag{5.6}$$

is a symmetry of the PDE

$$u_{tt} = (u_{x_1 x_1})^{-1/3} + \sum_{k=2}^{n} u_{x_k x_k}.$$

Both symmetries (5.5) and (5.6) form cyclic groups of order 2 if $\epsilon = 1$ and of order 4 if $\epsilon = -1$.

Employment of the point transformations (2.12) that relate equations (5.1) and (5.2) lead us to only one result, which is the trivial transformation

$$x'_1 = t \qquad x'_i = x_i \qquad t' = x_1 \qquad u' = u \qquad i = 2, 3, \ldots, n \tag{5.7}$$

that maps the PDE

$$u'_{t't'} = u'_{x'_1 x'_1} + \sum_{k=2}^{n} F_k(u'_{x'_k x'_k})$$

to the PDE

$$u_{tt} = u_{x_1 x_1} - \sum_{k=2}^{n} F_k(u_{x_k x_k})$$

where the functions F_k are arbitrary.

6. Remarks

Point transformations of a restricted class have been introduced here. A complete classification of such transformations admitted by specific n-dimensional nonlinear equations is presented. It should pointed out that transformations of more general classes admitted by the same equations exist. For example, the transformation

$$x_1' = \sqrt{2}x_1 + x_2 \qquad x_2' = x_1 + \sqrt{2}x_2 \qquad x_i' = x_i$$
$$t' = t \qquad u' = u \qquad i = 3, 4, \ldots, n$$

is a point symmetry of the PDE

$$u_{tt} = [F_1(u)u_{x_1}]_{x_1} - [F_1(u)u_{x_2}]_{x_2} + \sum_{k=3}^{n}[F_k(u)u_{x_k}]_{x_k}$$

where the functions $F_i(u)$ are arbitrary. A second example is the point transformation

$$x_1' = x_1 + 2x_2 + \frac{2}{\sqrt{3}}x_3 \qquad x_2' = 2x_1 + x_2 + \frac{2}{\sqrt{3}}x_3$$
$$x_3' = \frac{2}{\sqrt{3}}x_1 + \frac{2}{\sqrt{3}}x_2 + x_3 \qquad t' = t \qquad u' = u$$

which maps the three-dimensional wave equation

$$u'_{t't'} = [F(u')u'_{x_1'}]_{x_1'} + [F(u')u'_{x_2'}]_{x_2'} - \tfrac{1}{3}[F(u')u'_{x_3'}]_{x_3'}$$

in the equation

$$u_{tt} = [F(u)u_{x_1}]_{x_1} + [F(u)u_{x_2}]_{x_2} - 3[F(u)u_{x_3}]_{x_3}.$$

Nevertheless the problem of determining the complete set of transformations of the class (2.1) for equations (1.1)–(1.3) is a very difficult task.

Although a restricted class of point transformations was considered in this paper, the analysis goes beyond Lie group analysis. In addition to Lie symmetries, we have obtained equivalence transformations, discrete symmetries and transformations relating different equations but of the same class. This shows that there is merit in studying point transformations directly in finite form with the ultimate dual goal of finding the complete set of point symmetries of PDEs and discovering new links between different equations, even though this analysis is more difficult than searching for Lie infinitesimal groups of transformations.

We point out that *all* the point transformations derived in this paper map a linear PDE to a linear PDE or a nonlinear PDE to a nonlinear PDE. This is also the case for the one-dimensional equations of (1.1)–(1.3) [8]. Furthermore, for a general class of PDEs $u_{tt} = H$, where H is a function of x, t, u and derivatives of u w.r.t. x, it can be shown [9] that no point transformation exists to map a nonlinear PDE of this class to a linear PDE of the same class and vice versa. It appears that this also applies to the n-dimensional equations (1.1)–(1.3). Non-local transformations that connect an n-dimensional linear and a nonlinear PDE may exist. For example, in the one-dimensional case, it is known that the Legendre (contact) transformation

$$x' = u_t \qquad t' = u_x \qquad u' = tu_t + xu_x - u \tag{6.1}$$

relates the linear wave equation

$$u'_{t't'} = F(t')u'_{x'x'}$$

and the nonlinear wave equation

$$u_{tt} = F(u_x)u_{xx}$$

provided that $u_{tt}u_{xx} - u_{xt}^2 \neq 0$. Transformation (6.1) forms a cyclic group of order 2 ($u'_{x'} = t, u'_{t'} = x$). For further study, such transformations may be considered for higher-dimensional equations.

The consideration of n-dimensional equations makes the paper have mathematical significance rather than physical. However, the results for the one-, two- and three-dimensional equations are special cases of the results that we have obtained. These nonlinear wave equations are well known for their physical applications. See, e.g., [1, 2] and the references therein. The results for the one-dimensional equations have been presented in [3–5, 8], while for the two- and three-dimensional equations, only the Lie analysis has been carried out [1]. The rest of the point transformations, which can be obtained as special cases of the n-dimensional equations, are new.

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